Discrete breathers (DB) and intrinsic localized modes (ILM) are synonymous dynamical states for nonlinear lattice wave equations - periodic in time and localized in space, and widely observed in many applications. I will start to review the main theoretical and experimental results on DB/ILM properties. I will highlight several hard problems - quantization, and impact on (non)equilibrium dynamics - which still await detailed analysis.

I will then discuss the DB/ILM legacy and impact on understanding a number of related phenomena. The Fermi-Pasta-Ulam model, which is the base of the famous FPU paradox, allows for exact qBreather solutions, which are periodic in time and localized in momentum space. Compact DB/ILM solutions exist for classes of lattice equations with global gauge symmetry and additional constraints based on destructive interference, e.g. for flat band lattices which admit compact states even at the linear limit. Such finetuned lattices with all bands flat at the linear limit may also admit exact traveling DB/ILM solutions.

DB/ILM solutions are integrable limit solutions for a class of nonlinear lattice wave equations, where they turn into single site excitations of uncoupled oscillator sets. The corresponding preserved actions lead to a set of observables off the integrable limit. They define equilibrium Poincare manifolds which account for the time an initial wave configuration needs to reach equipartition. The manifolds further account for the statistics of nonergodic fluctuations at equilibrium – a long-sought technique to quantitatively measure the impact of DB/ILM solutions on (non)equilibrium fluctuations by correlating individual microscopic dynamical space-time events with statistical properties of nonergodic fluctuations.